

# ON THE KOSZUL COHOMOLOGY OF CANONICAL AND PRYM-CANONICAL BINARY CURVES

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**ABSTRACT.** In this paper we study Koszul cohomology and the Green and Prym-Green conjectures for canonical and Prym-canonical binary curves. We prove that if property  $N_p$  holds for a canonical or a Prym-canonical binary curve of genus  $g$  then it holds for a generic canonical or Prym-canonical binary curve of genus  $g + 1$ . We also verify the Green and Prym-Green conjectures for generic canonical and Prym-canonical binary curves of low genus ( $6 \leq g \leq 15$ ,  $g \neq 8$  for Prym-canonical and  $3 \leq g \leq 12$  for canonical).

## 1. INTRODUCTION

Let  $C$  be a smooth curve,  $L$  a line bundle and  $\mathcal{F}$  a coherent sheaf on  $C$ . We recall that the Koszul cohomology group  $K_{p,q}(C, \mathcal{F}, L)$  is the middle term cohomology of the complex:

$$(1) \quad \Lambda^{p+1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q-1}) \xrightarrow{d_{p+1,q-1}} \Lambda^p H^0(L) \otimes H^0(\mathcal{F} \otimes L^q) \xrightarrow{d_{p,q}} \Lambda^{p-1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q+1})$$

where

$$d_{p,q}(s_1 \wedge \dots \wedge s_p \otimes u) := \sum_{l=1}^p (-1)^l s_1 \wedge \dots \wedge \hat{s}_l \wedge \dots \wedge s_p \otimes (s_l u).$$

If  $\mathcal{F} = \mathcal{O}_C$  the groups  $K_{p,q}(C, \mathcal{O}_C, L)$  are denoted by  $K_{p,q}(C, L)$ . The Koszul cohomology theory has been introduced in [11] and has been extensively studied in particular in the case of the canonical bundle. We recall that Green and Lazarsfeld ([11]) proved that for any smooth curve  $C$  of genus  $g$  and Clifford index  $c$ ,  $K_{g-c-2,1}(C, K_C) \neq 0$ . Green's conjecture says that this result is sharp i.e.  $K_{p,1}(C, K_C) = 0$  for all  $p \geq g - c - 1$ . The Clifford index for a general curve is  $\lfloor \frac{g-1}{2} \rfloor$ , so generic Green's conjecture says that  $K_{p,1}(C, K_C) = 0$  for all  $p \geq \lfloor \frac{g}{2} \rfloor$ , or equivalently, by duality,  $K_{p,2}(C, K_C) = 0$ , i.e. property  $N_p$  holds, for all  $p \leq \lfloor \frac{g-3}{2} \rfloor$ . Generic Green's conjecture has been proved by Voisin in [20],[21]. Green's conjecture has also been verified for curves of odd genus and maximal Clifford index ([21], [12]), for general curves of given gonality ([20], [18] [17]), for curves on  $K3$ -surfaces ([20], [21], [1]), and in other cases (see [2]).

Another interesting case is when the line bundle is Prym canonical,  $L = K_C \otimes A$  where  $A$  is a non trivial 2 torsion line bundle. This case has been studied in [8], where the Prym-Green conjecture has been stated. This is an analogue of the Green conjecture for general curves, namely it says that for a general Prym-canonical curve  $(C, K_C \otimes A)$ , we have  $K_{p,2}(C, K_C \otimes A) = 0$ , i.e. property  $N_p$  holds, for all  $p \leq \lfloor \frac{g}{2} - 3 \rfloor$ . Prop.3.1 of [8] shows that for any  $(C, K_C \otimes A)$  and  $p > \lfloor \frac{g}{2} - 3 \rfloor$ ,  $K_{p,2}(C, K_C \otimes A) \neq 0$ .

Debarre in [7] proved that a generic Prym-canonical curve of genus  $g \geq 6$  is projectively normal (property  $N_0$ ) and for  $g \geq 9$  its ideal is generated by quadrics (property  $N_1$ ).

In [5] the Prym-Green conjecture is proved for genus  $g = 10, 12, 14$  by degeneration to irreducible nodal curves and computation with Macaulay2. In a private communication Gavril Farkas told us that they could verify the conjecture also for  $g = 18, 20$ . The computations made in [5] for genus 8 and 16 suggest that the Prym-Green conjecture may be false for genus which is a multiple of 8 or perhaps a power of 2. The possible failure of the Prym-Green conjecture in genus 8 is extensively discussed in the last section of [5], where a geometric interpretation of this phenomenon is given.

In this paper we study Koszul cohomology and the Green and Prym-Green conjectures for canonical and Prym-canonical binary curves. Recall that a binary curve of genus  $g$  is a stable curve consisting of two rational components  $C_j$ ,  $j = 1, 2$  meeting transversally at  $g + 1$  points. The canonical and Prym-canonical models of binary curves that we analyze are the one used in [3] and [6] and described

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in the next section. The main result of the paper (Theorem (3.2)) says that if property  $N_p$  holds for a Prym-canonical binary curve of genus  $g$  then it holds for a generic Prym-canonical binary curve of genus  $g + 1$ . In particular, if the Prym-Green conjecture is true for a Prym-canonical binary curve of genus  $g = 2k$ , then it is true for a general Prym-canonical binary curve of genus  $g = 2k + 1$ .

Moreover we verify the conjecture by a direct computation for  $g = 6, 9, 10, 12, 14$  (see Corollary (3.4)).

As a consequence, we show that the generic Prym-canonical curve of genus  $g$  satisfies property  $N_0$  for  $g \geq 6$ , property  $N_1$  for  $g \geq 9$  (already shown by Debarre), property  $N_2$  for  $g \geq 10$ , property  $N_3$  for  $g \geq 12$  and property  $N_4$  for  $g \geq 14$  (Corollary (3.4)).

For  $g = 8$  and  $g = 16$  our computations on Prym-canonical binary curves also suggest that Prym-Green conjecture's might fail, in fact in our examples we find that  $K_{\frac{g}{2}-3,2}(C, K_C \otimes A) = 1$  both for  $g = 8$  and  $g = 16$  (see Remark (3.5)).

An analogue result of Theorem (3.2) is proven for canonically embedded binary curves (Theorem (4.2)), where we show that if property  $N_p$  holds for a canonical binary curve of genus  $g$ , then the same property holds for a general canonical binary curve of genus  $g + 1$ . In particular, if the Green conjecture is true for a canonical binary curve of genus  $g = 2k - 1$ , then it is true for a general canonical binary curve of genus  $g = 2k$ .

Theorem (3.2) and analogue computations with maple in genus  $g = 3, 5, 7, 9, 11$ , imply that for a general canonical binary curve, if  $g \geq 3$ , then property  $N_0$  holds (see also [3] section 2), if  $g \geq 5$ , then property  $N_1$  holds, if  $g \geq 7$ , then property  $N_2$  holds, if  $g \geq 9$ , then property  $N_3$  holds, and if  $g \geq 11$ , then property  $N_4$  holds.

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## 2. CANONICAL AND PRYM-CANONICAL BINARY CURVES

**2.1. Construction of canonical binary curves.** Recall that a binary curve of genus  $g$  is a stable curve consisting of two rational components  $C_j$ ,  $j = 1, 2$  meeting transversally at  $g + 1$  points. Moreover,  $H^0(C, \omega_C)$  has dimension  $g$  and the restriction of  $\omega_C$  to the component  $C_j$  is  $K_{C_j}(D_j)$  where  $D_j$  is the divisor of nodes on  $C_j$ . Since  $K_{C_j}(D_j) \cong \mathcal{O}_{\mathbb{P}^1}(g - 1)$  we observe that the components are embedded by the complete linear system  $|\mathcal{O}_{\mathbb{P}^1}(g - 1)|$  in  $\mathbb{P}^{g-1}$ .

Following [3], we assume that the first  $g$  nodes are  $P_i = (0, \dots, 0, 1, 0, \dots, 0)$ , with 1 at the  $i$ -th place,  $i = 1, \dots, g$ . Then we can assume that  $C_j$  is the image of the map

$$(2) \quad \begin{aligned} \phi_j : \mathbb{P}^1 &\rightarrow \mathbb{P}^{g-1}, \quad j = 1, 2 \\ \phi_j(t, u) &:= \left[ \frac{M_j(t, u)}{(t - a_{1,j}u)}, \dots, \frac{M_j(t, u)}{(t - a_{g-1,j}u)} \right] \end{aligned}$$

with  $M_j(t, u) := \prod_{l=1}^g (t - a_{l,j}u)$ ,  $j = 1, 2$  and  $\phi_j([a_{l,j}, 1]) = P_l$ ,  $l = 1, \dots, g$ .

We see that the remaining node is the point  $P_{g+1} := [1, \dots, 1]$  and it is the image of  $[1, 0]$  through the maps  $\phi_j$ ,  $j = 1, 2$ . One can easily check that, for generic values of the  $a_{i,j}$ 's,  $C = C_1 \cup C_2$  is a canonically embedded binary curve.

**2.2. Construction of Prym-canonical binary curves.** Let  $C$  be a binary curve of genus  $g$ , and  $A \in \text{Pic}^0(C)$  a nontrivial line bundle. Then  $H^0(C, \omega_C \otimes A)$  has dimension  $g - 1$  and the restriction of  $\omega_C \otimes A$  to the component  $C_j$  is  $K_{C_j}(D_j)$  where  $D_j$  is the divisor of nodes on  $C_j$ . Since  $K_{C_j}(D_j) \cong \mathcal{O}_{\mathbb{P}^1}(g - 1)$ , the components are embedded by a linear subsystem of  $\mathcal{O}_{\mathbb{P}^1}(g - 1)$ , hence they are projections from a point of rational normal curves in  $\mathbb{P}^{g-1}$ . Viceversa, let us take 2 rational curves embedded in  $\mathbb{P}^{g-2}$  by non complete linear systems of degree  $g - 1$  intersecting transversally at  $g + 1$  points. Then their union  $C$  is a binary curve of genus  $g$  embedded either by a linear subsystem of  $\omega_C$  or by a complete linear system  $|\omega_C \otimes A|$ , where  $A \in \text{Pic}^0(C)$  is nontrivial (see e.g. [4], Lemma 10). In [6] (Lemma 3.1) we constructed a binary curve  $C$  embedded in  $\mathbb{P}^{g-2}$  by a linear system  $|\omega_C \otimes A|$  with  $A^{\otimes 2} \cong \mathcal{O}_C$ , and

$A$  is non trivial. Let us now recall this construction and denote a binary curve with this embedding a Prym-canonical binary curve.

Assume that the first  $g-1$  nodes, are  $P_i = (0, \dots, 0, 1, 0, \dots, 0)$  with 1 at the  $i$ -th place,  $i = 1, \dots, g-1$ , the remaining two nodes are  $P_g := [t_1, \dots, t_{g-1}]$  with  $t_i = 0$  for  $i = 1, \dots, [\frac{g}{2}]$ ,  $t_i = 1$ , for  $i = [\frac{g}{2}] + 1, \dots, g-1$ . and  $P_{g+1} := [s_1, \dots, s_{g-1}]$  with  $s_i = 1$  for  $i = 1, \dots, [\frac{g}{2}]$ ,  $s_i = 0$ , for  $i = [\frac{g}{2}] + 1, \dots, g-1$ .

Then the component  $C_j$  is the image of the map

$$(3) \quad \begin{aligned} \phi_j : \mathbb{P}^1 &\rightarrow \mathbb{P}^{g-2}, \quad j = 1, 2, \text{ where} \\ \phi_1(t, u) &:= \left[ \frac{tM_1(t, u)}{(t - a_{1,1}u)}, \dots, \frac{tM_1(t, u)}{(t - a_{k,1}u)}, \frac{-M_1(t, u)d_1a_{k+1,1}u}{A_1(t - a_{k+1,1}u)}, \dots, \frac{-M_1(t, u)d_1a_{g-1,1}u}{A_1(t - a_{g-1,1}u)} \right] \\ \phi_2(t, u) &:= \left[ \frac{tM_2(t, u)}{(t - a_{1,2}u)}, \dots, \frac{tM_2(t, u)}{(t - a_{k,2}u)}, \frac{-M_2(t, u)d_2a_{k+1,2}u}{A_2(t - a_{k+1,2}u)}, \dots, \frac{-M_2(t, u)d_2a_{g-1,2}u}{A_2(t - a_{g-1,2}u)} \right] \end{aligned}$$

with  $k := [\frac{g}{2}]$ ,  $M_j(t, u) := \prod_{r=1}^{g-1} (t - a_{r,j}u)$ , and  $A_j = \prod_{i=1}^{g-1} a_{i,j}$ ,  $j = 1, 2$ ,  $d_2$  is a nonzero constant and  $d_1 = \frac{-d_2 A_1}{A_2}$ . Notice that we have  $\phi_j([a_{l,j}, 1]) = P_l$ ,  $l = 1, \dots, g-1$ ,  $\phi_j([0, 1]) = P_g$ ,  $\phi_j([1, 0]) = P_{g+1}$ ,  $j = 1, 2$ . In Lemma 3.1 of [6] we proved that for a general choice of  $a_{i,j}$ 's,  $C = C_1 \cup C_2$  is a binary curve embedded in  $\mathbb{P}^{g-2}$  by a linear system  $|\omega_C \otimes A|$  with  $A^{\otimes 2} \cong \mathcal{O}_C$  and  $A$  nontrivial. In fact, recall that  $Pic^0(C) \cong \mathbb{C}^{*g} \cong \mathbb{C}^{*g+1}/\mathbb{C}^*$ , where  $\mathbb{C}^*$  acts diagonally, and in Lemma 3.1 of [6] it is shown and our line bundle  $A$  corresponds to the element  $[(h_1, \dots, h_{g+1})] \in \mathbb{C}^{*g+1}/\mathbb{C}^*$ , where  $h_i = 1$ , for  $i < [\frac{g}{2}] + 1$ ,  $h_i = -1$ , for  $i = [\frac{g}{2}] + 1, \dots, g-1$ ,  $h_g = -1$ ,  $h_{g+1} = 1$ , so in particular  $A$  is of 2-torsion.

### 3. PROPERTY $N_p$ FOR PRYM-CANONICAL BINARY CURVES

Let  $C \subset \mathbb{P}^{g-2}$  be a Prym-canonical binary curve embedded by  $\omega_C \otimes A$ , with  $A^{\otimes 2} \cong \mathcal{O}_C$ , as in (3). In this section we study the Koszul cohomology for these curves, in particular we investigate property  $N_p$ , i.e. the vanishing of  $K_{p,2}(C, K_C \otimes A)$ . Since by duality (prop. 1.4 [9]) we have  $K_{p,2}(C, K_C \otimes A) \cong K_{g-3-p,0}(C, K_C, K_C \otimes A)^\vee$ , this vanishing is equivalent to the injectivity of the Koszul map

$$(4) \quad F_{g-3-p} : \Lambda^{g-3-p} H^0(C, \omega_C \otimes A) \otimes H^0(C, \omega_C) \rightarrow \Lambda^{g-4-p} H^0(C, \omega_C \otimes A) \otimes H^0(C, \omega_C^2 \otimes A).$$

Our strategy is to compare this map with analogous Koszul maps for a partial normalization of the curve  $C$  at one node and possibly use induction on the genus.

To this end, let us introduce some notation: set  $k := [\frac{g}{2}]$  and denote by  $\tilde{C}_r$  the partial normalization of  $C$  at the node  $P_r$  with  $r \leq k$  if  $g = 2k$ ,  $r \geq k+1$  if  $g = 2k+1$ . This choice of the node is necessary in order to obtain the Prym-canonical model for the curve  $\tilde{C}_r$ . In fact, observe that in this way, for a general choice of the  $a_{i,j}$ 's, the projection from  $P_r$  sends the curve  $C$  to the Prym-canonical model of  $\tilde{C}_r$  in  $\mathbb{P}^{g-3}$  given by the line bundle  $K_{\tilde{C}_r} \otimes A'_r$  where  $A'_r$  corresponds to the point  $(h'_1, \dots, h'_{g-1}, 1) \in \mathbb{C}^{*g}/\mathbb{C}^*$ , with  $h'_i = 1$  for  $i \leq [\frac{g-1}{2}]$ ,  $h'_i = -1$  for  $i = [\frac{g-1}{2}] + 1, \dots, g-1$ , as described above. In fact  $(\tilde{C}_r, A'_r)$  is parametrized by  $a'_{i,j} = a_{i,j}$  for  $i \leq r-1$ ,  $j = 1, 2$ ,  $a'_{i,j} = a_{i+1,j}$  for  $i \geq r$ ,  $j = 1, 2$ . So if we set  $d'_j := \frac{d_j}{a_{r,j}}$ ,  $j = 1, 2$ , we clearly have a pair  $(\tilde{C}_r, A'_r)$  as in (3). For simplicity let us choose  $d_2 = 1$ , so  $d_1 = -\frac{A_1}{A_2}$ , hence  $d'_2 := \frac{1}{a_{r,2}}$ ,  $d'_1 := -\frac{A_1}{A_2 a_{r,1}}$ .

To simplify the notation, set  $T_g := H^0(C, \omega_C \otimes A)$ ,  $H_g := H^0(C, \omega_C)$ ,  $B_g := H^0(C, \omega_C^2 \otimes A)$ . Denote by  $\{t_1, \dots, t_{g-1}\}$  the basis of  $T_g$  given by the coordinate hyperplane sections in  $\mathbb{P}^{g-2} \cong \mathbb{P}(T_g^\vee)$  and by  $\{s_1, \dots, s_g\}$  the basis of  $H_g$  given by the coordinate hyperplane sections in  $\mathbb{P}^{g-1} \cong \mathbb{P}(H_g^\vee)$ .  $T_{g-1,r} := H^0(\tilde{C}_r, \omega_{\tilde{C}_r} \otimes A'_r)$ ,  $H_{g-1,r} := H^0(\tilde{C}_r, \omega_{\tilde{C}_r})$ ,  $B_{g-1,r} := H^0(\tilde{C}_r, \omega_{\tilde{C}_r}^2 \otimes A'_r)$ . Denote by  $\{t'_1, \dots, t'_{g-2}\}$  the basis of  $T_{g-1,r}$  given by the coordinate hyperplane sections in  $\mathbb{P}^{g-3} \cong \mathbb{P}(T_{g-1,r}^\vee)$  and by  $\{s'_1, \dots, s'_{g-1}\}$  the basis of  $H_{g-1,r}$  given by the coordinate hyperplane sections in  $\mathbb{P}^{g-2} \cong \mathbb{P}(H_{g-1,r}^\vee)$ .

We have the following injections:

$$(5) \quad T_{g-1,r} \xrightarrow{I_r} T_g, \quad t'_i \mapsto t_i \text{ for } i \leq r-1, \quad t'_i \mapsto t_{i+1} \text{ for } i \geq r,$$

$$(6) \quad H_{g-1,r} \xrightarrow{J_r} H_g, \quad s'_i \mapsto s_i \text{ for } i \leq r-1, \quad s'_i \mapsto s_{i+1} \text{ for } i \geq r.$$

Clearly these maps induce an injective map

$$(7) \quad B_{g-1,r} \xrightarrow{L_r} B_g,$$

which on the set of generators of  $B_{g-1,r}$  given by  $t'_i s'_j$ ,  $i = 1, \dots, g-2$ ,  $j = 1, \dots, g-1$  is given by  $t'_i s'_j \mapsto I_r(t'_i) J_r(s'_j)$ . We claim that this map is well defined and injective by the definition of the  $t'_i$ 's and  $s'_j$ 's. In fact the restriction of  $\sum \alpha_{i,j} t'_i s'_j$  to the two rational components of  $\tilde{C}_r$  yields two polynomials  $Q_1$  and  $Q_2$ . On the other hand we have  $(\sum \alpha_{i,j} I_r(t'_i) J_r(s'_j))|_{C_i} = (t - a_{r,i})^2 Q_i$ , hence  $L_r$  is well defined and injective. We finally have a map

$$(8) \quad \Lambda^{l-1} T_{g-1,r} \xrightarrow{\wedge t_r} \Lambda^l T_g,$$

where by  $\wedge t_r$  we indicate the composition of the natural map induced by  $I_r$  at the level of the  $(l-1)$ -th exterior power  $\Lambda^{l-1} T_{g-1,r} \rightarrow \Lambda^{l-1} T_g$  composed by the wedge product with  $t_r$ ,  $\Lambda^{l-1} T_g \xrightarrow{\wedge t_r} \Lambda^l T_g$ .

As in (4), denote by  $F_l : \Lambda^l T_g \otimes H_g \rightarrow \Lambda^{l-1} T_g \otimes B_g$  the Koszul map.

We have the following commutative diagram

$$(9) \quad \begin{array}{ccccc} \Lambda^l T_g \otimes H_g & \xrightarrow{F_l} & \Lambda^{l-1} T_g \otimes B_g & \xrightarrow{\pi_r} & \langle t_r \rangle \wedge \Lambda^{l-2} T_g \otimes B_g \\ \uparrow \wedge t_r \otimes J_r & & & \nearrow \wedge t_r \otimes L_r & \\ \Lambda^{l-1} T_{g-1,r} \otimes H_{g-1,r} & \xrightarrow{\tilde{F}_{l-1}} & \Lambda^{l-2} T_{g-1,r} \otimes B_{g-1,r} & & \end{array}$$

From now on, given a multi-index  $I = (i_1, \dots, i_l)$  we denote by  $t_I := t_{i_1} \wedge \dots \wedge t_{i_l}$ .

To study the injectivity of the maps  $F_l$ , a preliminary reduction comes from the following

**Lemma 3.1.** *Let  $W \subset \Lambda^l T_g \otimes H_g$  be the subspace generated by the elements of the form  $t_I \otimes s_j$ , where  $j \notin I$ . The map  $F_l : \Lambda^l T_g \otimes H_g \rightarrow \Lambda^{l-1} T_g \otimes B_g$  is injective if and only if  $F_l|_W$  is injective.*

*Proof.* Assume that  $v \in \Lambda^l T_g \otimes H_g$ ,  $v = \sum_{I, |I|=l} \sum_{j=1 \dots g} \lambda_j^I t_I \otimes s_j$  is such that  $F_l(v) = 0$ .  $F_l(v) = \sum_{J, |J|=l-1} \sum_{I=J \cup \{m\}} \sum_{j=1 \dots g} \lambda_j^I \epsilon(I, J) t_J \otimes t_m s_j = 0$ , where  $\epsilon(I, J) = \pm 1$ , depending on the position of  $m$  in the multi-index  $I = J \cup \{m\}$ . Then if we fix a multi-index  $J$  with  $|J| = l-1$ , we must have  $\sum_m \sum_{j=1 \dots g} \lambda_j^{J \cup \{m\}} \epsilon(J \cup \{m\}, J) t_J \otimes t_m s_j = 0$  and therefore

$$\sigma_J := \sum_m \sum_{j=1 \dots g} \lambda_j^{J \cup \{m\}} \epsilon(J \cup \{m\}, J) t_m s_j = 0.$$

So we have  $\sigma_J|_{C_1} \equiv 0$ , namely, if we denote by  $P_1(t) := t \cdot M_1(t, 1)$ , as in (3), we have

$$\begin{aligned} & \sum_{j=1 \dots g} \sum_{m \leq k} \lambda_j^{J \cup \{m\}} \epsilon(J \cup \{m\}, J) \frac{P_1(t)}{t - a_{m,1}} \frac{P_1(t)}{t - a_{j,1}} \\ & + \sum_{j=1 \dots g} \sum_{m \geq k+1} \lambda_j^{J \cup \{m\}} \epsilon(J \cup \{m\}, J) \frac{P_1(t) a_{m,1}}{A_2 t (t - a_{m,1})} \frac{P_1(t)}{t - a_{j,1}} = 0 \end{aligned}$$

If we evaluate in  $t = a_{m,1}$ , there remains only one term in the sum, namely the one with  $j = m$ , and hence we have

$$\begin{aligned} & \lambda_m^{J \cup \{m\}} \epsilon(J \cup \{m\}, J) a_{m,1}^2 \cdot \prod_{r \neq m, r=1 \dots g-1} (a_{m,1} - a_{r,1})^2 = 0, \text{ if } m \leq k, \\ & \lambda_m^{J \cup \{m\}} \epsilon(J \cup \{m\}, J) \frac{a_{m,1}^2}{A_2} \cdot \prod_{r \neq m, r=1 \dots g-1} (a_{m,1} - a_{r,1})^2 = 0, \text{ if } m \geq k+1, \end{aligned}$$

hence we have  $\lambda_m^{J \cup \{m\}} = 0$  for all  $m$ .

Since this holds for every multi-index  $J$  of cardinality  $l-1$ , we have shown that we can write  $v = \sum_{I, |I|=l} \sum_{j=1 \dots g, j \notin I} \lambda_j^I t_I \otimes s_j$ .  $\square$

We can now state and prove our main result.

**Theorem 3.2.** *Assume that  $g = 2k$ , or  $g = 2k+1$  and take an integer  $p \leq k-3$ . If property  $N_p$  holds for a binary curve  $\tilde{C}$  of genus  $g-1$  embedded in  $\mathbb{P}^{g-3}$  by  $|\omega_{\tilde{C}} \otimes A'|$  as in (3) for a generic choice of the parameters  $a'_{i,j}$ , then it holds for all binary curves  $C$  of genus  $g$  embedded in  $\mathbb{P}^{g-2}$  by  $|\omega_C \otimes A|$  as in (3) for a generic choice of the  $a_{i,j}$ 's.*

*Proof.* We want to prove that  $K_{p,2}(C, K_C \otimes A) = 0$  for a binary curve of genus  $g$  and we know that  $K_{p,2}(\tilde{C}_r, K_{\tilde{C}_r} \otimes A'_r) = 0$ , for the curve  $\tilde{C}_r$  which is obtained from  $C$  by projection from  $P_r$  with  $r \geq k+1$  if  $g = 2k+1$ ,  $r \leq k$  if  $g = 2k$ .

By duality,  $K_{p,2}(C, K_C \otimes A) \cong K_{g-3-p,0}(C, K_C, K_C \otimes A)^\vee$ , so the statement is equivalent to prove injectivity of the Koszul map

$$F_{g-3-p} : \Lambda^{g-3-p} T_g \otimes H_g \rightarrow \Lambda^{g-4-p} T_g \otimes B_g.$$

By assumption we know injectivity of the map

$$\tilde{F}_{g-4-p} : \Lambda^{g-4-p} T_{g-1,r} \otimes H_{g-1,r} \rightarrow \Lambda^{g-5-p} T_{g-1,r} \otimes B_{g-1,r}.$$

For simplicity let us denote by  $l := g-3-p$ .

Assume first of all that  $g = 2k+1$  and consider the projection of  $C$  from  $P_{g-1}$ .

Recall that by Lemma (4.1) we can reduce to prove injectivity of  $F_l$  restricted the subspace  $W$  generated by such  $t_I \otimes s_j$  with  $j \notin I$ . Note that we can decompose  $W$  as  $W := X_{g-1} \oplus Y_{g-1}$ , where  $X_{g-1}$  is the intersection with  $W$  of the image of the map  $\wedge t_{g-1} \otimes J_{g-1}$  in diagram (9) and  $Y_{g-1}$  is the subspace of  $W$  generated by such  $t_I \otimes s_j$  with  $g-1 \notin I$  and  $j \notin I$ :

$$X_{g-1} = \langle t_{g-1} \wedge t_J \otimes s_j \mid j \notin J \rangle, \quad Y_{g-1} = \langle t_I \otimes s_j \mid j, g-1 \notin I \rangle.$$

Assume now that  $F_l(x_{g-1} + y_{g-1}) = 0$ , where  $x_{g-1} \in X_{g-1}$ ,  $y_{g-1} \in Y_{g-1}$ . Then we have  $0 = \pi_{g-1} \circ F_l(x_{g-1} + y_{g-1}) = \pi_{g-1} \circ F_l(x_{g-1}) = (\wedge t_{g-1} \otimes L_{g-1}) \circ \tilde{F}_{l-1}(x_{g-1})$ , by the commutativity of diagram (9). Hence  $x_{g-1} = 0$ , since by induction we are assuming that  $\tilde{F}_{l-1}$  is injective. So we have reduced to prove injectivity of  $F_l$  restricted to  $Y_{g-1}$ .

Now consider the projection of  $C$  from the point  $P_{g-2}$ .

Set

$$Y'_{g-2} = \langle t'_J \otimes s'_j \mid j, g-2 \notin J \rangle \subset \Lambda^{l-1} T_{g-1,g-2} \otimes H_{g-1,g-2}$$

Observe that the image  $X_{g-2} := (\wedge t_{g-2} \otimes J_{g-2})(Y'_{g-2})$  is contained in  $Y_{g-1}$  and in fact

$$X_{g-2} = \langle t_{g-2} \wedge t_J \otimes s_j \mid j, g-1 \notin J \rangle.$$

So we have  $Y_{g-1} = X_{g-2} \oplus Y_{g-2}$ , where  $Y_{g-2}$  is the subspace of  $Y_{g-1}$  generated by those elements of the form  $t_I \otimes s_j$  where  $g-2, g-1, j \notin I$ . We have the following commutative diagram

$$(10) \quad \begin{array}{ccccc} \Lambda^l Y_{g-1} & \xrightarrow{F_l} & \Lambda^{l-1} T_g \otimes B_g & \xrightarrow{\pi_{g-2}} & \langle t_{g-2} \rangle \wedge \Lambda^{l-2} T_g \otimes B_g \\ \uparrow \wedge t_{g-2} \otimes J_{g-2} & & & \nearrow \wedge t_{g-2} \otimes L_{g-2} & \\ \Lambda^{l-1} Y'_{g-2} & \xrightarrow{\tilde{F}_{l-1}} & \Lambda^{l-2} T_{g-1,g-2} \otimes B_{g-1,g-2} & & \end{array}$$

Assume that  $v = x_{g-2} + y_{g-2} \in Y_{g-1} = X_{g-2} \oplus Y_{g-2}$  is such that  $F_l(v) = 0$ , then we have  $0 = \pi_{g-2} \circ F_l(x_{g-2} + y_{g-2}) = \pi_{g-2} \circ F_l(x_{g-2})$ . So  $0 = \tilde{F}_{l-1}(x_{g-2})$  by the commutativity of the diagram, and this implies  $x_{g-2} = 0$  by induction. Therefore we can assume that  $v \in Y_{g-2}$ , hence  $v$  is a linear combination of vectors of the form  $t_I \otimes s_j$  where  $g-2, g-1, j \notin I$ .

Repeat the procedure, i.e. project from the points  $P_r$ ,  $r = g - 3 \dots l$ . This can be done since  $l = g - 3 - p \geq k + 1$ . In this way we can reduce to prove injectivity for the restriction of the map  $F_l$  to the subspace  $Y_l$  of  $W$  generated by the elements of the form  $t_I \otimes s_j$  where  $l, \dots, g - 1, j \notin I$ . Observe that since  $|I| = l$ , we have  $Y_l = 0$ , so  $F_l$  is injective and the theorem is proved.

If  $g = 2k$  the proof is analogous: we subsequently project from the points  $P_1, P_2, \dots, P_{g-l}$ . As before note that this can be done since  $g - l = p + 3 \leq k$ . In this way we reduce to prove injectivity for the restriction of the map  $F_l$  to the subspace  $Y$  of  $W$  generated by the elements of the form  $t_I \otimes s_j$  where  $1, 2, 3, \dots, g - l, j \notin I$  and since  $|I| = l$ , we have  $Y = 0$ , so  $F_l$  is injective and the theorem is proved.  $\square$

**Corollary 3.3.** *If the Prym-Green conjecture is true for a semi-canonical binary curve of genus  $g = 2k$  as in (3), then it is true for a Prym-canonical binary curve of genus  $g = 2k + 1$  as in (3) for generic parameters  $a_{i,j}$ .*

*Proof.* The conjecture for  $g = 2k + 1$  says that  $K_{k+1,0}(C, K_C, K_C \otimes A) = 0$ , or analogously that property  $N_{k-3}$  holds for a generic  $C$  embedded with  $K_C \otimes A$ . Hence the corollary immediately follows from Theorem (3.2) with  $i = k - 3$ .  $\square$

**Corollary 3.4.** *The generic Prym-canonical curve of genus  $g$  satisfies property  $N_0$  for  $g \geq 6$ ,  $N_1$  for  $g \geq 9$ ,  $N_2$  for  $g \geq 10$ ,  $N_3$  for  $g \geq 12$ ,  $N_4$  for  $g \geq 14$ .*

*Proof.* With a direct computation one verifies the Prym-Green conjecture for explicit examples of Prym-canonical binary curves as in (3) for  $g = 6, 9, 10, 12, 14$ , so the proof follows from Theorem (3.2) for generic semicanonical binary curves, and then by semicontinuity for generic Prym-canonical smooth curves.

To do the computations we wrote a very simple maple code (<http://www-dimat.unipv.it/~frediani/prym-can>) in which we explicitly give the matrix representing the Koszul map  $F_l$ : for every multi-index  $J$  with  $|J| = l - 1$ , we take the projection of the image of  $F_l$  onto  $t_J \otimes B_g$  and we restrict it to the rational components  $C_j$ . So we have two polynomials in one variable and we take their coefficients.

Once the matrix is constructed, for  $g = 6, 9, 10, 12$ , maple computed its rank modulo 131, which turned out to be maximal. In the case  $g = 14$  the order of the matrices was too big, so Riccardo Murri made the rank computation using the Linbox ([14]) and Rheinfall ([16]) free software libraries. Two different rank computation algorithms were used: Linbox' "black box" implementation of the block Wiedemann method ([13, 19]), and Rheinfall's Gaussian Elimination code([15]). Results obtained by either method agree.

In both cases, the GNU GMP library ([10]) provided the underlying arbitrary-precision representation of rational numbers and exact arithmetic operations.

$\square$

**Remark 3.5.** *For Prym-canonical curves of genus 8, the maple computation on specific examples of binary curves gives  $\dim K_{1,2}(C, K_C \otimes A) = 1$ . This result is compatible with the computations in [5].*

*For Prym-canonical binary curves of genus 16, we constructed the matrix representing the Koszul map  $F_8$  on examples using maple and Riccardo Murri computed its rank as explained in the proof of Corollary (3.4). Again it turned out that  $\dim K_{5,2}(C, K_C \otimes A) = 1$ , confirming the computations in [5].*

#### 4. PROPERTY $N_p$ FOR CANONICAL BINARY CURVES

In analogy with the Prym-canonical case, we study now property  $N_p$  for canonical binary curves with the same inductive method, projecting from a node. So, let  $C \subset \mathbb{P}^{g-1}$  be a canonical binary curve and denote by  $\tilde{C}_r$  the partial normalization of  $C$  at the node  $P_r$ ,  $1 \leq r \leq g$ . As above, for a general choice of the  $a_{i,j}$ 's, the projection from  $P_r$  sends the curve  $C$  to the canonical model of  $\tilde{C}_r$  in  $\mathbb{P}^{g-2}$ , where  $\tilde{C}_r$  is parametrized by  $a'_{i,j} = a_{i,j}$  for  $i \leq r - 1$ ,  $j = 1, 2$ ,  $a'_{i,j} = a_{i+1,j}$  for  $i \geq r$ ,  $j = 1, 2$ .

Set  $H_g := H^0(C, \omega_C)$ ,  $D_g := H^0(C, \omega_C^2)$ ,  $F_l : \Lambda^l H_g \otimes H_g \rightarrow \Lambda^{l-1} H_g \otimes D_g$  the Koszul map. Denote as before by  $\{s_1, \dots, s_g\}$  the basis of  $H_g$  given by the coordinate hyperplane sections in  $\mathbb{P}^{g-1} \cong \mathbb{P}(H_g^\vee)$ .

$H_{g-1,r} := H^0(\tilde{C}_r, \omega_{\tilde{C}_r})$ ,  $D_{g-1,r} := H^0(\tilde{C}_r, \omega_{\tilde{C}_r}^2)$ . Denote by  $\{s'_1, \dots, s'_{g-1}\}$  the basis of  $H_{g-1,r}$  given by the coordinate hyperplane sections in  $\mathbb{P}^{g-2} \cong \mathbb{P}(H_{g-1,r}^\vee)$ .

We have the injections  $H_{g-1,r} \xrightarrow{J_r} H_g$ , as in (6) and  $D_{g-1,r} \xrightarrow{L_r} D_g$ , which on the set of generators of  $B_{g-1,r}$  given by  $s'_i s'_j$ ,  $i, j = 1 \dots g-1$ , is given by  $s'_i s'_j \mapsto J_r(s'_i) J_r(s'_j)$ .

We finally have a map

$$(11) \quad \Lambda^{l-1} H_{g-1,r} \xrightarrow{\wedge s_r} \Lambda^l H_g,$$

where by  $\wedge s_r$  we indicate the composition of the natural map induced by  $J$  at the level of the  $l-1$ -th exterior power  $\Lambda^{l-1} H_{g-1,r} \rightarrow \Lambda^{l-1} H_g$  composed by the wedge product with  $s_r$ ,  $\Lambda^{l-1} H_g \xrightarrow{\wedge s_r} \Lambda^l H_g$ .

We are interested in property  $N_p$  for these curves, hence by duality, in the vanishing of  $K_{g-2-p,1}(C, K_C)$ . Clearly the vanishing of  $K_{l,1}(C, K_C)$  is equivalent to the injectivity of the map

$$(12) \quad \frac{\Lambda^l H_g \otimes H_g}{\Lambda^{l+1} H_g} \rightarrow \Lambda^{l-1} H_g \otimes D_g$$

coming from the Koszul complex.

Notice that there is an isomorphism between  $\frac{\Lambda^l H_g \otimes H_g}{\Lambda^{l+1} H_g}$  and the subspace  $V_g$  of  $\Lambda^l H_g \otimes H_g$  generated by the elements of the form  $s_I \otimes s_j$ , where  $j \geq i_1$ , so the above injectivity is equivalent to the injectivity of the restriction of  $F_l$  to  $V_g$ .

We have the following commutative diagram

$$(13) \quad \begin{array}{ccccc} V_g & \xrightarrow{F_l} & \Lambda^{l-1} H_g \otimes D_g & \xrightarrow{\pi_r} & \langle s_r \rangle \wedge \Lambda^{l-2} H_g \otimes D_g \\ \uparrow \wedge s_r \otimes J_r & & & \nearrow \wedge s_r \otimes L_r & \\ V_{g-1,r} & \xrightarrow{\tilde{F}_{l-1}} & \Lambda^{l-2} H_{g-1,r} \otimes D_{g-1,r} & & \end{array}$$

where  $V_{g-1,r}$  is the subspace of  $\Lambda^{l-1} H_{g-1,r} \otimes H_{g-1,r}$  generated by the elements of the form  $s'_j \otimes s'_j$ , where  $j \geq j_1$ .

Let  $W \subset V_{g,l}$  be the subspace generated by the elements of the form  $s_I \otimes s_j$ , where  $j \notin I$  and  $j \geq i_1$ .

**Remark 4.1.** *The map  $F_l : V_g \rightarrow \Lambda^{l-1} \otimes B_g$  is injective if and only if  $F_l|_W$  is injective.*

*Proof.* The proof is completely analogous to the proof of (3.1).  $\square$

**Theorem 4.2.** *If property  $N_p$  holds for a canonical binary curve of genus  $g-1$  as in (4.2), then the same property holds for a canonical binary curve of genus  $g$  as in (4.2) for a generic choice of the parameters.*

*Proof.* From the above discussion we know that the statement is equivalent to prove injectivity of the Koszul map  $F_l : V_g \rightarrow \Lambda^{l-1} H_g \otimes D_g$  for  $l = g-2-p$ , while by assumption we know injectivity of the map  $\tilde{F}_{l-1} : V_{g-1,r} \rightarrow \Lambda^{l-2} H_{g-1,r} \otimes D_{g-1,r}$ .

We first project from  $P_g$ . By Remark (4.1) we can reduce to prove injectivity of  $F_l$  restricted the subspace  $W$  generated by such  $s_I \otimes s_j$  with  $j \notin I, j > i_1$ . Note that as before we can decompose  $W$  as  $W := X_g \oplus Y_g$ , where  $X_g$  is the intersection with  $W$  of the image of the map  $\wedge s_g \otimes J_g$  in diagram (13) and  $Y_g$  is the subspace of  $W$  generated by such  $s_I \otimes s_j$  with  $g \notin I$  and  $j \notin I, j > i_1$ :

$$X_g = \langle s_g \wedge s_J \otimes s_j \mid j \notin J, j > j_1 \rangle, \quad Y_g = \langle s_I \otimes s_j \mid j, g \notin I, j > i_1 \rangle$$

If  $F_l(x_g + y_g) = 0$ , where  $x_g \in X_g$ ,  $y_g \in Y_g$ , then  $0 = \pi_g \circ F_l(x_g + y_g) = \pi_g \circ F_l(x_g) = (\wedge s_g \otimes L_g) \circ \tilde{F}_{l-1}(x_g)$ . Hence  $x_g = 0$ , since by induction  $\tilde{F}_{l-1}$  is injective. So we have reduced to prove injectivity of  $F_l$  restricted to  $Y_g$ .

Repeat the procedure, i.e. project from the points  $P_r$ ,  $r = g - 1 \dots l$ . In this way we can reduce to prove injectivity for the restriction of the map  $F_l$  to the subspace  $Y_l$  of  $W$  generated by the elements of the form  $s_I \otimes s_j$  where  $l, \dots, g, j \notin I, j > i_1$ . Observe that since  $|I| = l$ , we have  $Y_l = 0$ , so  $F_l$  is injective and the theorem is proved.  $\square$

**Remark 4.3.** Notice that, by the theorem of Green and Lazarsfeld ([11]), if  $p > g - \lfloor \frac{g}{2} \rfloor - 2$ , condition  $N_p$  does not hold for any curve  $\tilde{C}$  of genus  $g - 1$ .

**Corollary 4.4.** If the Green conjecture is true for a canonical binary curve of genus  $g = 2k - 1$  as in (4.2), then it is true for a canonical binary curve of genus  $g = 2k$  as in (4.2) for a generic choice of the parameters.

*Proof.* The conjecture for  $g = 2k$  says that  $K_{k,1}(C, K_C) = 0$ , or analogously that property  $N_{k-2}$  holds for  $C$  embedded with  $K_C$ . By assumption we know that  $K_{k-1,1}(\tilde{C}, K_{\tilde{C}}) = 0$ , namely that property  $N_{k-2}$  holds for  $\tilde{C}$  embedded with  $K_{\tilde{C}}$ , so the thesis immediately follows from (4.2).  $\square$

With maple (<http://www.dimat.unipv.it/~frediani/greenfinal.tar.gz>) one verifies the conjecture for  $g = 5, 7, 9, 11$ , so one can prove with the same method that if  $g \geq 3$ , then property  $N_0$  holds (see also [3] section 2), if  $g \geq 5$ , then property  $N_1$  holds, if  $g \geq 7$ , then property  $N_2$  holds, and if  $g \geq 9$ , then property  $N_3$  holds, and if  $g \geq 11$ , then property  $N_4$  holds.

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